Ordinary Annuities

Future Value of an Ordinary Annuity

- **Annuity (basic definition) →** is a sequence of payments (usually equal), dispersed or received, at equal intervals of time.

Payment Intervals → These are tied to a compound interest rate, and generally the payment and interest intervals match.
e.g. a home mortgage is paid monthly using a rate that is compounded monthly.

Periodic payment for an annuity or Periodic Rent → Using the word *rent* → indicates that the rent we pay for a house or apartment is also an annuity

Term of an Annuity (life of an annuity) → Runs from the beginning of the first rent period to the end of the last rent period.

Future Value of an Ordinary Annuity

Annuities → Ways of Classification:

1) *The first classification deals with the **Term of the Annuity:***

- **Annuities Certain** → begin and end at a set point in time.

e.g. *mortgage.*

- **Contigent Annuities** → have a beginning or ending date that depends on some event.

e.g. *Payment structure of retirement funds, life annuities and life insurance is tipically a contigent annuity because the end of the payments depends on an event*.

- **Perpetuity** → annuity with a specific starting time but an infinite number of payments.
  Provide periodic income from a sum of money without using any of the principal.

2) *The second classification deals with the placement of the **Periodic Rent:***

- **Ordinary Annuity (annuity immediate)** →places the payments at the end of each rent period.
  (term used by the actuarial sciences and insurance industry)

- **Annuity Due** →places the payments at the beginning of each rent period.
  (term used by the business and banking industry)

From a different perspective these annuities are the same sequence of payments but are just valued at different points in time. → **Neither of these terms is especially descriptive of the actual situation.**

Annuity - immediate

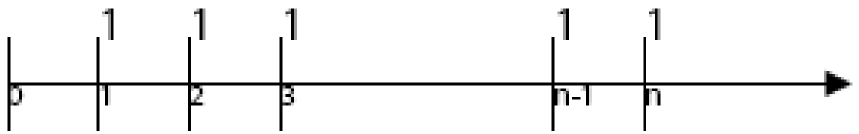

Figure 1: Time diagram for annuity-immediate

Present value:

$$a_{\overline{n}|} = v + v^2 + \dots v^n = \frac{1 - v^n}{i} \qquad (1)$$

Accumulated value at time n:

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i} = \frac{v^{-n} - 1}{i} \qquad (2)$$

Here i is the effective annual rate of interest, $v = \frac{1}{1+i}$ the discount factor. When the rate of interest should be indicated, we use symbols $a_{\overline{n}|i}$ and $s_{\overline{n}|i}$ for $a_{\overline{n}|}$ and $s_{\overline{n}|}$.

Annuity - due

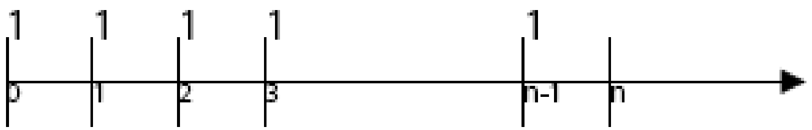

Figure 2: Time diagram for annuity-due

Present value:

$$\ddot{a}_{\overline{n}|} = 1 + v + \ldots v^{n-1} = \frac{1 - v^n}{d}$$

Accumulated value at time n:

$$\ddot{s}_{\overline{n}|} = \frac{v^{-n} - 1}{d}$$

Future Value of an Ordinary Annuity

Annuities → Ways of Classification:

3) *The third classification depends on the alignment of the* ***Compound Interest Conversion Periods*** *and* ***Payment Intervals:***

- **Simple Annuity** →with a simple annuity the interest is compounded at the same frequency as the payments are made.
- **General Annuity** → with a general annuity the payments and conversion periods do not align. The interest conversions may occur more or less often than the payments.

Besides the Periodic Rent, every annuity also has a ***Present Value*** or a ***Future Value*** or both.

Located at the beginning of the term for loans or mortgages
It makes sense because:
- the borrower receives this "bundle" of money at the start of the loan.
- the payments occur in the months that follow.

Also goes by the term amount (or accumulated value);
Located at the end of an annuity's term.
Important for:
- savings or retirement accounts → here the saver makes periodic payments that acumulate and earn interest until some future date where the "bundle" is located.

Future Value of an Ordinary Annuity

Summary of Basic Definitions with Notation

***i* Interest Rate/Period**	*nominal rate divided by periods per year,* $i = \frac{i(m)}{m}$.
***n* Number of payments**	*number of payments in the term of the annuity.*
***R* Payment**	*euro value of the periodic payment or rent.*
***S_n* Future Value**	*sum of all the payments valued at the date of the last payment.*

Amount of an Ordinary Annuity (Future Value): $S_n = \frac{R\left[1-(1+i)^n\right]}{i}$

Compact Notation for the Future Value: $S_n = Rs_n^i$

Amount of 1 per period ← s_n^i

Present Value of an Ordinary Annuity

- ***<u>Present Value (Discounted Value) of an ordinary annuity (Annuity Immediate):</u>***

(definition) → sum of money at the beginning of the term that is equivalent to the sequence of payments that follow.

Any loan repaid with a sequence of payments requires a calculation that uses the Present Value of an annuity.

Present value
(money borrowed)

Periodic Rent
(payments made on the loan)

Present Value Calculations

Typically, with a loan the amount of money to be borrowed is known and the payment is calculated for a given rate term → Ocasionally the borrower knows the size of the payment he can afford and then must calculate the present value to find out how much he can borrow.

Other applications: income annuities, asset evaluation, the sale of coupon bonds, capital budgeting and insurance premiums.

Approaches to find the formula:

1) Find the sum of the present values of the individual periodic payments of the annuity using geometric series.
2) Find the present value of the amount of annuity as expressed by the future value formula.

Present Value of an Ordinary Annuity

Summary of Basic Definitions with Notation

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***n* Number of payments**	*number of payments in the term of the annuity.*
***R* Payment**	*euro value of the periodic payment or rent.*
A_n Present Value	*sum of the present values of all payments.*

Present Value for an Ordinary Annuity: $A_n = \frac{R\left[1-(1+i)^{-n}\right]}{i}$

Compact Notation for the Present Value: $A_n = Ra_n^i$

Present worth for the Present Value ← a_n^i

The Periodic Payment or Rent for an Ordinary Annuity

- **We often know the present value of a loan or the amount of a savings program and we want to find the periodic payment (or Rent).**

Given the number of loans transacted in the world of finance

Finding the payment may be the single most repeated financial calculation.

Clue words indicate that a Problem requires the bundle of money to be at the **Present** or at the **Future.**

Clue words used:
loan, cash value, now, estate, money to produce income and selling price

Clue words used:
amount, future value, accumulate, save, balance and settlement

Approach to find R:

- Compute the value of the annuity and accumulation factor
- Solve for *R*

$$s_{n}^{i} = \left[\frac{(1+i)^{n}-1}{i}\right]$$

Payment :

For the same amount of money: $\frac{1}{a_{n}^{i}} = \frac{1}{s_{n}^{i}} + i$

The Number of Payments or the Term of an Ordinary Annuity

Whenever a sum of money is invested to provide a regular income, the number of payments it will provide must be calculated.

Retirement accounts, insurance settlements, estates, and the sale of property can all provide regular income.

If the income continues without ending → it is a *Perpetuity* (since a perpetuity pays out only interest and never the principal.

The income will last only so long if the recipient needs a level of income that requires the use of the principal.

Situations which require calculating the Number of Payments:

1) Knowing the present value and the size of the withdrawals → find out the number of withdrawals (payments) and the smaller last payment when the account is closed once the money is gone.

2) Knowing the highest affordable payment and the target sum → find out the number of required payments.

The Interest Rate of an Ordinary Annuity

Knowing the rate of an annuity provides a realistic way to compare different investment or loan rates.

Many lending agencies advertise sums that can be borrowed along with the required payment and term, but they do not bother to tell their interest rates.

Rates of return on a simple interest or discount basis are quite useful in making judgments about investments and capital expenditures.

Since many investments give periodic returns that are Annuities

It will be valuable to discern the rate of return for making financial decisions

Method to find i:

- In the Future Value factor denominator. $S_n^i = \left[\frac{(1+i)^n - 1}{i}\right]$ *i* is both the exponential term and the

The only simple method of solving for *i* → Linear Interpolation.

Truth in Lending and the Annual Percentage Rate

Money lenders have historically tried to hide the true rate that they are charging for lending money.

Many types of institutions lend money and their lending practices vary considerably.

e.g. we can borrow money from the cash value of life insurance or from banks, retail stores, or credit cards.

There need to be a way to make reasonable comparisons of their lending rates.

Annual percentage Rate (APR)
(stated on the loan papers or other contract documents issued by the lender)

(Definition) The *annual percentage* rate (APR) is the rate at which the cash value of the loan equals the present value of the payments